SNEWS Word Problems

Note to teachers: I have not been explicit with the "significant figures" in these calculations. Astronomical measurements are frequently imprecise: we might be happy to measure a star's mass to within five solar masses, for example. In constructing these problems, I have opted to stress concepts, rather than arithmetic. I would advise grading on a partial credit system, giving several points to a student if he or she can manipulate the necessary equation in the proper way, even if the arithmetic in the end has a mistake or two. If "sig figs" are a large part of your curriculum (as I know they were in my day), you may wish to grade them more heavily.

1 Exercise Solutions

Problem 1: *How Big is a Neutrino Detector?*

(a) We can find the volume with a little dimensional analysis, a good exercise in scientific notation:

$$\begin{aligned} 50 \text{ kilotons} &= \frac{10^6 \text{ kg}}{1 \text{ kiloton}} \times \frac{1 \text{ L}}{1 \text{ kg}} \times \frac{1 \text{ m}^3}{10^3 \text{ L}} \\ &= 5 \times 10^1 \times 10^6 \times 10^{-3} \text{ m}^3 \\ &= 5 \times 10^4 \text{ m}^3. \end{aligned}$$

"To a first approximation", as the scientists say, we guess that the tank is cubic. Therefore, to find the length of its side, we take the cube root:

$$\boxed{a = \sqrt[3]{5 \times 10^4 \text{ m}^3} \approx 36.8 \text{ m.}} \tag{1}$$

Problem 2: *How Strong is the Signal?*

All the parts of this problem can be solved using the given equation for intensity:

$$I(R) = I_0 \left(\frac{R_0}{R}\right)^2. \tag{2}$$

(a) Here, we're told that $I_0 = 1\frac{\mathrm{J}}{\mathrm{m^2 s}}$, that $R_0 = 10\mathrm{m}$ and that we wish to know the intensity at $R = 30\mathrm{m}$. Plug the distances we know into the equation:

$$\begin{aligned} I(30\text{ m}) &= I_0 \left(\frac{10\text{ m}}{30\text{ m}}\right)^2 \\ &= I_0 \left(\frac{1}{3}\right)^2 \\ &= \frac{I_0}{9} \\ &= \frac{1}{9}\frac{\mathrm{J}}{\mathrm{m^2 s}}, \end{aligned}$$

or in decimal form,

$$\boxed{I(30\text{ m}) \approx 0.11\frac{\mathrm{J}}{\mathrm{m^2 s}}.} \tag{3}$$

(b) This part may be a little trickier, because we're not told what I_0 is. However, knowing algebra means we don't *have* to know I_0. What we are told is that

$$I(R) = \frac{1}{100} I_0. \tag{4}$$

We can plug this information into the intensity equation, like so:

$$I(R) = I_0 \left(\frac{R_0}{R}\right)^2 = \frac{1}{100} I_0. \tag{5}$$

Whatever I_0 is, it is surely a number greater than zero (otherwise, the supernova would be invisible!). Therefore, we can divide both sides by the unknown quantity:

$$\left(\frac{R_0}{R}\right)^2 = \frac{1}{100}. \tag{6}$$

Taking the square root of both sides, we find that

$$\frac{R_0}{R} = \frac{1}{10}. \tag{7}$$

Cross-multiplying now shows that

$$\boxed{R = 10R_0 = 10\text{ megaparsecs.}} \tag{8}$$

Problem 3: *Temperature Scales*

The key formula here is

$$K = C + 273.15 = \frac{F + 459.67}{1.8}. \tag{9}$$

(a) Well, there's no guarantee exactly what you find a comfortable room temperature, but lots of people seem to agree on around 72°F. In kelvins, this is

$$\frac{72 + 459.67}{1.8} \approx 295.4K. \tag{10}$$

Subtract 273.15 K to get the Celsius equivalent temperature, about 22°C.

(b) Again, just subtract 273.15 K:

$$5500°C - 273.15°C = 5226.85\ K. \tag{11}$$

Problem 4: *Stellar Masses*

Over the last several decades, scientists have discovered that the mass of a star is the single most important factor in determining how long it will "live" and in what fashion it will "die". Frequently, masses for distant stars are given as multiples of our Sun's mass.

(a) First, we multiply the mass of the sun (given in the data table) by 15:

$$15M_{\text{Sun}} \approx 2.98 \times 10^{31}\ \text{kg}. \tag{12}$$

This answers part (i); to solve part (ii), divide the previous result by the mass of the Earth:

$$2.98 \times 10^{31}\text{kg} \times \frac{1M_{\text{Earth}}}{5.9736 \times 10^{34}\ \text{kg}} \approx 5.0 \times 10^{6} M_{\text{Earth}}. \tag{13}$$

(b) 70 Jupiter masses is about 1.33×10^{29} kg. Dividing this by M_{Sun} (the same value we used in part (a)) shows that 70 Jupiter masses is about 0.067 solar masses.

Problem 6: *Half-Lives*

Here, all the problems can be solved using the half-life decay equation,

$$N(t) = N_0 \cdot \left(\frac{1}{2}\right)^{t/\tau}. \tag{14}$$

It may be a useful hint to know that $N(t)$ and N_0 can be in any units we wish: grams, atoms, moles, etc. If we measure N_0 in grams, $N(t)$ will be in grams. Likewise, as long as t and τ are in the same units—seconds, years, or what have you—we don't need to be picky about what units to use. It is not necessary to convert everything to seconds first!

(a) *Radium decay.* We're told that the decay process for radium obeys the rule

$$^{226}\text{Ra} \rightarrow ^{4}\text{He} + ^{222}\text{Rn}. \tag{15}$$

Here, $\tau = 1602$ years, and we're told that the time period t is 1000 years. Any decent calculator can handle the tough work:

$$N(t) = 10 \text{ g} \cdot \left(\frac{1}{2}\right)^{\frac{1000}{1602}} \approx 6.49 \text{ g}. \tag{16}$$

Radioactive decay "burns up" 3.51 g of radium, leaving 6.49 g behind. (Remember, this process takes a thousand years!) A little chemistry can tell us how much radon is released. We know that every atom of radium which decays releases one atom of radon, and the Periodic Table informs us that one mole of radium has a mass of 226.0254 grams. Furthermore, at "standard temperature and pressure" (273.15 K and 1 atmosphere), one mole of any gas takes up 22.4 liters of space. Using these facts, a little dimensional analysis shows that

$$3.51 \text{ g Ra} \cdot \frac{1 \text{ mol Ra}}{226.0254 \text{ g Ra}} \cdot \frac{1 \text{ mol Rn}}{1 \text{ mol Ra}} \cdot \frac{22.4 \text{ L Rn}}{1 \text{ mol Rn}} = 0.348 \text{ L Rn}. \tag{17}$$

(b) *Carbon-14 dating.* Here, we're given that $\tau = 5,730$ years. The "tricky" part is that we don't know N_0 *or* $N(t)$, but (like the intensity problem earlier) we know their ratio. The archaeologist measures that

$$N(t) = 0.80 N_0. \tag{18}$$

Substituting this into the half-life equation, we find that

$$0.80 N_0 = N_0 \left(\frac{1}{2}\right)^{\frac{t}{\tau}}. \tag{19}$$

Just like in the intensity problem, we can divide both sides by N_0:

$$0.80 = \left(\frac{1}{2}\right)^{\frac{t}{\tau}}. \tag{20}$$

We can solve for t/τ using logarithms:

$$\frac{t}{\tau} = \log_{\frac{1}{2}} 0.80. \tag{21}$$

Multiplying both sides by τ now shows that

$$t = \tau \log_{\frac{1}{2}} 0.80, \tag{22}$$

which works out to about 1845 years. This is a little *younger* than the coins claimed to be, but remember that they could have been put in the box when they were several years old already.

What about the experimental error? The archaeologist could only determine the carbon-14 content to within 2%. Supposing that the real value were 78%, then the box would be

$$t = \tau \log_{\frac{1}{2}} 0.80 = 2054 \tag{23}$$

years old. Likewise, you can work out that if the real value were 82%, the box would only be 1640 years old. This gives an estimate of the error involved—and an indication of how tricky getting accurate measurements can be!

(c) *Supernova light curve.* We're told that nickel-56 has a half-life of 6.077 days, and again we're given the ratio of $N(t)$ and N_0. This time,

$$N(t) = \frac{1}{64} N_0. \tag{24}$$

Using the half-life equation,

$$\frac{1}{64} N_0 = N_0 \left(\frac{1}{2}\right)^{\frac{t}{\tau}}, \tag{25}$$

and dividing both sides by N_0,

$$\frac{1}{64} = \left(\frac{1}{2}\right)^{\frac{t}{\tau}}. \tag{26}$$

Here, it's helpful to know the powers of two. You can easily check that $2^6 = 64$. Turing this upside down, we see that

$$\left(\frac{1}{2}\right)^{6} = \frac{1}{64}. \tag{27}$$

Therefore, $\frac{t}{\tau} = 6$, and

$$\boxed{t \approx 36.5 \text{ days.}} \tag{28}$$

(d) *Bonus.* Since we just showed that in 36.5 days the nickel-56 would be down to $\frac{1}{64}$ of its original amount, if *all* the light came from nickel-56, then the supernova would only be $\frac{1}{64}$ as bright. Observing that it remains half its original brightness after 80 days means that some other elements are responsible.

Problem 7: *Einstein's Equation*

According to Einstein's Special Theory of Relativity, matter and energy are interchangeable. It is possible to convert an amount of mass into pure energy, which may take the form of light or other electromagnetic radiation. The exact rule is given by Einstein's famous equation,

$$E = mc^2. \tag{29}$$

Here, c is the speed of light, roughly 3×10^8 meters per second. If m is given in kilograms and c in meters per second, then E will have units of joules.

(a) Here, $m = 1$ kg. According to Einstein's equation, then,

$$\boxed{E = (1 \text{ kg}) \cdot \left(3 \times 10^8 \frac{\text{m}}{\text{s}}\right)^2 = 9 \times 10^{16} \text{J.}} \tag{30}$$

(b) Each photon should carry away half the energy of the original electron-positron pair, but since the electron and the positron have the same mass, each of the two photons should have the "mass-energy" of one electron. Here, it's helpful to use the fourth column of the particle-data table, which gives masses in terms of eV/c^2. The table tells us that an electron has a mass of $5.11 \times 10^5\ \mathrm{eV}/c^2$. By Einstein's equation, we would multiply this by c^2 to get the energy, but this just cancels the c^2 already there! Knowing that $1\mathrm{MeV} = 10^6\mathrm{eV}$, we can immediately say that the mass-energy of one electron is 0.511 MeV. Consequently, each photon has an energy of 0.511 MeV.

You can find the same result by using the mass in kilograms and multiplying by $c^2 = 9 \times 10^{16} \frac{\mathrm{m}^2}{\mathrm{s}^2}$. Remember that $1\ \mathrm{eV} = 1.602 \times 10^{-19}\mathrm{J}$.

(c) According to the quantum theory Max Planck helped found, the wavelength of a photon is inversely proportional to its energy. Typically, we use the Greek letter λ (*lambda*) to stand for the wavelength. Planck's equation says that

$$E = \frac{hc}{\lambda} \tag{31}$$

where h is *Planck's constant,* a number which experiments show is roughly 6.626×10^{-34} joule-seconds. There are (at least) two ways to solve this part: we can either plug the value of E we found earlier into Planck's equation, or we can combine Planck's equation with Einstein's. The two approaches should give identical answers.

To demonstrate the latter approach, first set the two formulas for E equal to each other:

$$mc^2 = \frac{hc}{\lambda}. \tag{32}$$

We can multiply both sides by λ to get

$$hc = \lambda mc^2, \tag{33}$$

and we can divide both sides by c to find

$$h = \lambda mc. \tag{34}$$

Moving the mc to the other side shows that

$$\lambda = \frac{h}{mc}. \tag{35}$$

Plugging in all the values we've been given, the wavelength works out to

$$\boxed{\lambda \approx 2.42 \times 10^{-12}\mathrm{m}.} \tag{36}$$

This is far too small see with the naked eye; in fact, it is a *gamma ray*, with a wavelength about 100 times smaller than the diameter of a hydrogen atom.

(d) Astrophysicists estimate that a Type II supernova can release 10^{44} joules of energy. We can turn this into a mass measurement by using Einstein's equation in reverse:

$$m = \frac{E}{c^2} \approx 1.11 \times 10^{27}\ \text{kg}. \tag{37}$$

We suppose that the original star had a mass roughly twenty times that of the Sun:

$$M = 20 M_{\text{Sun}} \approx 3.9782 \times 10^{31}\ \text{kg}. \tag{38}$$

This is much larger than the mass m which went into making the supernova. To see exactly how much larger, we compute the ratio:

$$\frac{m}{M} = \frac{1.11 \times 10^{27}\ \text{kg}}{3.9782 \times 10^{31}\ \text{kg}} \approx 2.79 \times 10^{-5}. \tag{39}$$

In everyday notation, the ratio is 0.0000279, or 0.00279%.